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Third-order structure function relations for quasi-geostrophic turbulence

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We derive two third-order structure function relations for quasi-geostrophic turbulence, one for the forward cascade of potential enstrophy and one for the inverse cascade of energy. These relations are the counterparts of Kolmovorov's (1941) four-fifths law for the third-order longitudinal structure functions of three-dimensional turbulence.

1. Introduction

Kolmogorov (1941) derived the third-order structure function relation for three-dimensional isotropic turbulence

$$\langle \delta u_L \delta u_L \delta u_L \rangle = -\tfrac{4}{5}\epsilon r, \tag{1.1}$$

where δu_L is the difference between the longitudinal velocity component at two points separated by the vector $\boldsymbol{r}$, ϵ is the mean dissipation rate of kinetic energy and $\langle \ldots \rangle$ designates a mean value. The longitudinal velocity component is the component in the direction of $\boldsymbol{r}$. Relation (1.1) is intimately connected with the notion of an energy flux from small to large wavenumbers in spectral space, that is a forward energy cascade. The correponding relation in Fourier space states that the spectral energy flux of kinetic energy from small to large wavenumbers is constant and equal to ϵ (see Frisch 1995).

In two-dimensional turbulence, there are two possible cascades (Kraichnan 1970): a forward cascade of enstrophy, defined as half the square of vorticity, and an inverse cascade of kinetic energy. For the forward enstrophy cascade the counterpart of (1.1) can be written as (Lindborg 1996)

$$\langle \delta u_L \delta \omega \delta \omega \rangle = -2\epsilon_\omega r, \tag{1.2}$$

where ω is the vorticity and ϵ_ω is the dissipation of enstrophy. Under the assumption of isotropy it can be shown that the following relations hold (Lindborg 1999):

$$\langle \delta u_L \delta \omega \delta \omega \rangle = -\nabla^2(\langle \delta u_L \delta u_L \delta u_L \rangle + \langle \delta u_L \delta u_T \delta u_T \rangle), \tag{1.3}$$

$$\langle \delta u_L \delta u_T \delta u_T \rangle = \frac{r}{3}\frac{\mathrm{d}}{\mathrm{d}r}\langle \delta u_L \delta u_L \delta u_L \rangle, \tag{1.4}$$

where T designates the transverse direction, which is perpendicular to $\boldsymbol{r}$. By using (1.3) and (1.4) relation (1.2) can be integrated to yield (Lindborg 1999)

$$\langle \delta u_L \delta u_L \delta u_L \rangle = \tfrac{1}{8}\epsilon_\omega r^3, \tag{1.5}$$

in the enstrophy inertial range of separation distances. Relation (1.5) has been reproduced in numerical simulations by Lindborg & Alvelius (2000) and Babiano & Dubos

(2005). For the two-dimensional inverse energy cascade the counterpart of Kolmogorov's relation (1.1) can be written as (Lindborg 1999)

$$\langle \delta u_L \delta u_L \delta u_L \rangle = \tfrac{3}{2} P r, \tag{1.6}$$

where P is the energy power input at small scales. This relation has been reproduced in numerical simulations by Bofetta, Celani & Vergassola (2000) and Babiano & Dubos (2005). In the two-dimensional inverse energy cascade range the third-order structure function is thus related to a small-scale energy source injecting energy into the system at a rate P, while in the three-dimensional forward cascade range it is related to a small-scale energy sink draining the system of energy at a rate ϵ. The minus sign in the three-dimensional relation (1.1) implies that energy is flowing in the direction from small to large wavenumbers in spectral space, while the plus sign in the two-dimensional relation (1.6) implies that energy is flowing in the opposite direction.

Quasi-geostrophic (QG) turbulence (Charney 1971) is similar to two-dimensional turbulence in that there are two inviscidly conserved quantities: potential enstrophy, defined as half the square of potential vorticity, and total energy. There are two possible cascade ranges corresponding to these two invariants: the forward potential enstrophy cascade range and the inverse energy cascade range. Just as in two-dimensional turbulence we would thus expect that there are two counterparts of Kolmogorov's relation: one for the forward enstrophy cascade and one for the inverse energy cascade. Nevertheless, QG turbulence is inherently three-dimensional, which has been demonstrated in numerical simulations by McWilliams, Weiss & Yavneh (1994). The ratio between the vertical and the horizontal length scale is typically of the order of f/N, where f is the Coriolis parameter and N is the Brunt–Väisälä frequency. This fact led Charney (1971) to introduce a stretched vertical coordinate $\tilde{z} = (N/f)z$, and the assumption of isotropy in the coordinates $x, y, \tilde{z}$, where x, y, z refer to a traditional Cartesian system. It may be thought that isotropy in this sense would imply that the QG counterpart of Kolmogorov's relation (1.1) should have a form where the left-hand side includes the three-dimensional longitudinal velocity, which generally has a component in the vertical direction, and where the right-hand side is linearly dependent on the three-dimensional separation distance. Kurien, Smith & Wingate (2006) have recently suggested that the QG counterpart of (1.1) in the potential enstrophy cascade range should have such a form. However, it can be argued that this cannot be the case. The third-order structure functions on the left-hand sides of (1.1), (1.2) and (1.6) originate from the nonlinear advective terms in the equations of motion. In the dynamic equation for QG turbulence the leading-order advective terms do not contain any vertical velocity component or any derivative with respect to the vertical. Therefore, the QG third-order structure function relations will look rather similar to the two-dimensional relations. In this paper, we shall briefly derive the QG relations.

2. Third-order structure function relations

2.1. *Forward potential enstrophy cascade range*

The inviscid QG equation for potential vorticity, q, can be written as

$$\frac{\partial q}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h q = 0, \tag{2.1}$$

where $\boldsymbol{u}_h$ is the horizontal velocity and ∇_h is the horizontal gradient operator. The dynamic equation for the two-point correlation, $\langle q q' \rangle$, of homogeneous QG turbulence can be derived from (2.1) by standard means (see standard textbooks like Batchelor 1953 or Monin & Yaglom 1975). A primed quantity should here be measured at a point separated by the vector $\boldsymbol{r}$ from the point where an unprimed quantity is measured. Spatial homogeneity implies that two-point correlations depend only on the separation vector $\boldsymbol{r}$. In the homogeneous case the inviscid dynamic equation for $\langle qq' \rangle$ is found to be

$$\frac{\partial}{\partial t}\langle qq' \rangle = \nabla_{h_r} \cdot \langle \boldsymbol{u}_h q q' - \boldsymbol{u}'_h q' q \rangle. \tag{2.2}$$

For QG turbulence we have

$$\nabla_h \cdot \boldsymbol{u}_h = 0. \tag{2.3}$$

Using this relation and the property of spatial homogeneity it is straightforward to rewrite (2.2) as

$$\frac{\partial}{\partial t}\langle qq \rangle - \frac{1}{2}\frac{\partial}{\partial t}\langle \delta q \delta q \rangle = \frac{1}{2}\nabla_{h_r} \cdot \langle \delta \boldsymbol{u}_h \delta q \delta q \rangle, \tag{2.4}$$

where $\delta q = q' - q$ and $\delta \boldsymbol{u}_h = \boldsymbol{u}'_h - \boldsymbol{u}_h$. We now assume that there is a small-scale potential enstrophy sink, which requires that there exists a dissipative force. Irrespective of the nature of this force the mean potential enstrophy equation can be written as

$$\frac{\partial}{\partial t}\left\langle \frac{qq}{2} \right\rangle = -\epsilon_q, \tag{2.5}$$

where ϵ_q is the dissipation rate of mean potential enstrophy. Substituting this expression into (2.4) and assuming that the turbulence is in a state of quasi-stationarity so that the second term on the left-hand side of (2.4) can be neglected, we find

$$\nabla_{h_r} \cdot \langle \delta \boldsymbol{u}_h \delta q \delta q \rangle = -4\epsilon_q. \tag{2.6}$$

This relation is expected to be valid in a range of separation distances which are sufficiently large for viscous forces to be negligible and at the same time sufficiently small for the assumption of quasi-stationarity to be valid. If this range is sufficiently broad it is justified to integrate the equation from zero horizontal separation out to a horizontal separation which is well inside this range. Assuming axisymmetry and introducing cylindrical coordinates we can integrate (2.6) to

$$\langle \delta u_\rho \delta q \delta q \rangle(\rho, r_z) = -2\epsilon_q \rho, \tag{2.7}$$

where $\rho = \sqrt{(x'-x)^2 + (y'-y)^2}$, $r_z = z' - z$ and δu_ρ is the velocity difference compontent in the same direction as the projection of $\boldsymbol{r}$ onto the horizontal plane. Here, r_z can be assumed to be measured in units which are stretched by a factor N/f. Relation (2.7) can be assumed to be approximately valid if the separation distance is considerably larger than a viscous length scale and at the same time if $\rho \ll L$ and $|r_z| \ll L$, where L is the large horizontal length scale of the quasi-geostrophic vortices, and where it should be remembered that r_z is measured in stretched units.

2.2. *Inverse energy cascade range*

To derive the third-order structure function relation for the inverse energy cascade we use the streamfunction formulation (Charney 1971) of the QG equations. Using Charney coordinates and omitting the tilde over the stretched vertical coordinate we can write

$$q = \nabla^2 \Psi, \tag{2.8}$$

$$\boldsymbol{u}_h = \boldsymbol{e}_z \times \nabla_h \Psi, \tag{2.9}$$

$$b = N \frac{\partial \Psi}{\partial z}, \tag{2.10}$$

where Ψ is the streamfunction, $\boldsymbol{e}_z$ is the vertical unit vector and b is the buoyancy, that is the normalized density fluctuation away from the mean stratification. The inviscid potential vorticity equation (2.1) can now be written as

$$\nabla \cdot \left(\frac{\partial \nabla \Psi}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \nabla \Psi \right) = 0, \tag{2.11}$$

from which follows that

$$\frac{\partial \nabla \Psi}{\partial t} + \boldsymbol{u}_h \cdot \nabla_h \nabla \Psi = \nabla \times \boldsymbol{\Phi}, \tag{2.12}$$

where $\boldsymbol{\Phi}$ is a vector potential which can be taken as divergence free without loss of generality. A Poisson equation for the vector potential can be obtained by taking the curl of (2.12), which eliminates the time derivative. Under the assumption of spatial homogeneity it is straightforward to derive the dynamic equation for the two-point correlation $\langle \nabla \Psi \cdot \nabla' \Psi' \rangle$:

$$\frac{\partial}{\partial t} \langle \nabla \Psi \cdot \nabla' \Psi' \rangle = \nabla_{h_r} \cdot \langle \boldsymbol{u}_h \nabla \Psi \cdot \nabla' \Psi' - \boldsymbol{u}'_h \nabla' \Psi' \cdot \nabla \Psi \rangle + \langle \nabla' \times \boldsymbol{\Phi}' \cdot \nabla \Psi + \nabla \times \boldsymbol{\Phi} \cdot \nabla' \Psi' \rangle. \tag{2.13}$$

The last term cancels by homogeneity,

$$\langle \nabla' \times \boldsymbol{\Phi}' \cdot \nabla \Psi + \nabla \times \boldsymbol{\Phi} \cdot \nabla' \Psi' \rangle = -\langle \boldsymbol{\Phi}' \cdot \nabla \times \nabla \Psi + \boldsymbol{\Phi} \cdot \nabla' \times \nabla' \Psi' \rangle = 0. \tag{2.14}$$

We now rewrite (2.13) in a similar way as we rewrote (2.2),

$$\frac{\partial}{\partial t} \langle \nabla \Psi \cdot \nabla \Psi \rangle - \frac{1}{2} \frac{\partial}{\partial t} \langle \delta \nabla \Psi \cdot \delta \nabla \Psi \rangle = \frac{1}{2} \nabla_{h_r} \cdot \langle \delta \boldsymbol{u}_h \delta \nabla \Psi \cdot \delta \nabla \Psi \rangle, \tag{2.15}$$

and introduce a small-scale force injecting energy into the system at a rate P. The mean energy equation can then be written as

$$\frac{\partial}{\partial t} \left\langle \frac{\nabla \Psi \cdot \nabla \Psi}{2} \right\rangle = P. \tag{2.16}$$

Substituiting this expression into (2.15) and neglecting the second term on the left-hand side we find

$$\nabla_{h_r} \cdot \langle \delta \boldsymbol{u}_h \delta \nabla \Psi \cdot \delta \nabla \Psi \rangle = 4P. \tag{2.17}$$

In a similar way as we integrated (2.6) we can integrate (2.17) to yield

$$\langle \delta u_\rho \delta \nabla \Psi \cdot \delta \nabla \Psi \rangle (\rho, r_z) = \langle \delta u_\rho \delta \boldsymbol{u}_h \cdot \delta \boldsymbol{u}_h \rangle (\rho, r_z) + \frac{1}{N^2} \langle \delta u_\rho \delta b \, \delta b \rangle (\rho, r_z) = 2P\rho. \tag{2.18}$$

This relation is expected to hold for separation distances which are larger than the characteristic length scale of the forcing and when $|r_z| \ll l$ and $\rho \ll l$, where l is the largest vertical length scale of the system measured in units which are stretched by a factor N/f. The first term on the left-hand side of the second equality in (2.18) represents the flux of kinetic energy from small to large scales and the second term represents the flux of potential energy. Charney (1971) hypothesized that there should be equipartition of energy between potential energy and the energy content in each of the two horizontal velocity components. If this hypothesis is extended to energy fluxes, the second term should be half the first term. Furthermore, for axisymmetric QG turbulence we have

$$\langle \delta u_\rho \delta u_\phi \delta u_\phi \rangle = \frac{\rho}{3} \frac{\partial}{\partial \rho} \langle \delta u_\rho \delta u_\rho \delta u_\rho \rangle, \tag{2.19}$$

a relation which can be derived in exactly in the same way as the corresponding two-dimensional relation (1.4) (see Lindborg 1999). Here, the subscript ϕ refers to a direction in the horizontal plane which is orthogonal to the direction indicated by the subscript ρ, just as in the traditional cylindrical coordinate formulation. Using this relation and the equipartition assumption we find

$$\langle \delta u_\rho \delta u_\rho \delta u_\rho \rangle(\rho, r_z) = P\rho, \tag{2.20}$$

$$\frac{1}{N^2} \langle \delta u_\rho \delta b \, \delta b \rangle(\rho, r_z) = \frac{2}{3} P\rho, \tag{2.21}$$

in the inverse energy cascade range of QG turbulence.

3. Conclusions

It is evident that the assumption of isotropy in the sense of Charney (1971) cannot be applied in the derivation of the third-order structure function relations for QG turbulence, since the advective terms of the QG equations do not contain any terms including the vertical velocity or the vertical differential operator. Even more generally, it can be argued that the concept of Charney isotropy has serious limitations. It can be given a reasonable interpretation when it is applied to scalar quantities, such as the second-order potential vorticity structure function $\langle \delta q \delta q \rangle$. For scalar quantities Charney isotropy means that they are functions only of $\tilde{r} = \sqrt{(x'-x)^2 + (y'-y)^2 + (\tilde{z}'-\tilde{z})^2}$. For tensor quantities, such as the second-order velocity structure function, $\langle \delta \boldsymbol{u} \delta \boldsymbol{u} \rangle$, the concept does not seem to be very fruitful. The vertical velocity is zero in QG turbulence and for this reason general velocity correlations cannot be invariant under rotations around non-vertical axes. Third-order structure functions formed out of the velocity vector and possibly some scalar quantity are third- or first-rank tensors and for this reason there is no fruitful way to apply the assumption of Charney isotropy to these quantities. In this paper, we have instead applied the assumption of axisymmetry to derive the third-order structure function relations (2.7) for the enstrophy cascade and (2.18) for the energy cascade. The analysis suggests that the theory of QG turbulence should be formulated under the constraint of axisymmetry rather than under the stronger constraint of isotropy.

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REFERENCES

Babiano, A. & Dubos, T. 2005 On the contribution of coherent vortices to the two-dimensional inverse energy cascade. *J. Fluid Mech.* **529**, 97–116.

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

Bofetta, G., Celani, A. & Vergassola, M. 2000 Inverse cascade in two-dimensional turbulence: deviations from Gaussianity. *Phys. Rev.* E **61**, R29–R32.

Charney, J. G. 1971 Geostrophic turbulence. *J. Atmos. Sci* **28**, 1087–1095.

Frisch, U. 1995 *Turbulence*. Cambridge University Press.

Kolmogorov, A. N. 1941 Dissipation of energy in the locally isotropic turbulence. *Dokl. Adad. Nauk SSSR* **32**(1). English transl. *Proc. R. Soc. Lond.* A **434** (1991), 15–17.

Kraichnan, R. H. 1970 Inertial-range transfer in two- and three-dimensional turbulence. *J. Fluid Mech.* **47**, 525–535.

Kurien, S., Smith, L. & Wingate, B. 2006 On the two-point correlation of potential vorticity in rotating and stratified turbulence. *J. Fluid Mech.* **555**, 131–140.

Lindborg, E. 1996 A note on Kolmovorov's third-order structure-function law, the local isotropy hypothesis and the pressure-velocity correlation. *J. Fluid. Mech.* **326**, 343–356.

Lindborg, E. 1999 Can the atmospheric kinetic energy be explained by two-dimensional turbulence? *J. Fluid. Mech.* **388**, 259–288.

Lindborg, E. & Alvelius, K. 2000 The kinetic energy spectrum of the two-dimensional enstrophy turbulence cascade. *Phys. Fluids* **12**, 945–947.

McWilliams, J. C., Weiss, J. B. & Yavneh, I. 1994 Anisotropy and coherent vortex structures in planetary turbulence. *Science* **264**, 410–413.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2. The MIT Press.